Solutions to short-answer questions

1 **a** $390^\circ = \dfrac{390 \times \pi}{180} = \dfrac{13\pi}{6}$

b $840^\circ = \dfrac{840 \times \pi}{180} = \dfrac{14\pi}{3}$

c $1110^\circ = \dfrac{1110 \times \pi}{180} = \dfrac{37\pi}{6}$

d $1035^\circ = \dfrac{1065 \times \pi}{180} = \dfrac{71\pi}{12}$

e $165^\circ = \dfrac{165 \times \pi}{180} = \dfrac{11\pi}{12}$

f $450^\circ = \dfrac{450 \times \pi}{180} = \dfrac{5\pi}{2}$

g $420^\circ = \dfrac{420 \times \pi}{180} = \dfrac{7\pi}{3}$

h $390^\circ = \dfrac{390 \times \pi}{180} = \dfrac{13\pi}{6}$

i $40^\circ = \dfrac{40 \times \pi}{180} = \dfrac{2\pi}{9}$

2 **a** $\dfrac{11\pi}{6} = \dfrac{11\pi \times 180}{6\pi} = 330^\circ$

b $\dfrac{17\pi}{4} = \dfrac{17\pi \times 180}{4\pi} = 765^\circ$

c $\dfrac{9\pi}{4} = \dfrac{9\pi \times 180}{4\pi} = 405^\circ$

d $\dfrac{7\pi}{12} = \dfrac{7\pi \times 180}{12\pi} = 105^\circ$

e $\dfrac{17\pi}{2} = \dfrac{17\pi \times 180}{2\pi} = 1530^\circ$

f $-\dfrac{11\pi}{4} = \dfrac{-11\pi \times 180}{4\pi} = -495^\circ$

g $-\dfrac{5\pi}{4} = \dfrac{-5\pi \times 180}{4\pi} = -225^\circ$

h $-\dfrac{13\pi}{4} = \dfrac{-13\pi \times 180}{4\pi} = -585^\circ$

i $\dfrac{23\pi}{4} = \dfrac{23\pi \times 180}{4\pi} = 1035^\circ$

3 **a**
$$\sin \frac{9\pi}{4} = \sin\left(2\pi + \frac{4\pi}{4}\right)$$
$$= \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

b $\cos\left(-\frac{5\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right)$

$= -\cos\frac{\pi}{4}$

$= -\frac{1}{\sqrt{2}}$

c $\sin\frac{3\pi}{2} = -1$

d $\cos -\frac{3\pi}{2} = 0$

e $\cos\frac{11\pi}{6} = \cos\left(2\pi - \frac{\pi}{6}\right)$

$= \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

f $\sin\frac{21\pi}{6} = \sin\left(4\pi - \frac{\pi}{6}\right)$

$= \sin -\frac{\pi}{2} = -1$

g $\tan -\frac{25\pi}{3} = -\tan\left(8\pi + \frac{\pi}{3}\right)$

$= -\tan\frac{\pi}{3} = -\sqrt{3}$

h $\tan -\frac{15\pi}{4} = -\tan\left(4\pi - \frac{\pi}{4}\right)$

$= -\tan -\frac{\pi}{4} = 1$

4 a $\text{Period} = \frac{2\pi}{\frac{1}{2}} = 4\pi$

$\text{Amplitude} = 4$

b $\text{Period} = \frac{2\pi}{6} = \frac{\pi}{3}$

$\text{Amplitude} = 5$

c $\text{Period} = \frac{2\pi}{4} = \frac{\pi}{2}$

$\text{Amplitude} = \frac{1}{3}$

d $\text{Period} = \frac{2\pi}{5}$

$\text{Amplitude} = 2$

e $\text{Period} = \frac{2\pi}{\frac{\pi}{4}} = 8$

$\text{Amplitude} = 7$

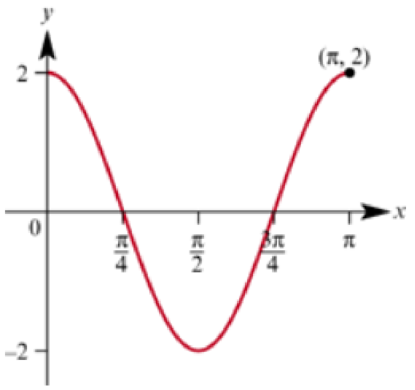
f Period $= \dfrac{2\pi}{\frac{2\pi}{3}} = 3$

Amplitude $= \dfrac{2}{3}$

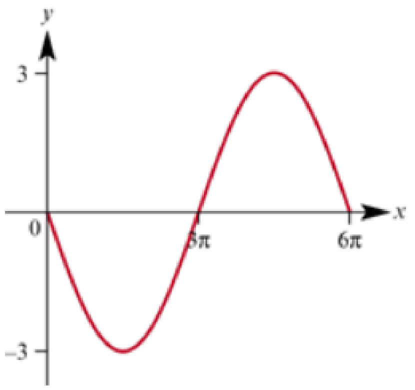
5 a Maximum when $\sin\theta = 1$, 5
Minimum when $\sin\theta = -1$, 1.

b Maximum when $\cos\theta = -1$, 9
Minimum when $\cos\theta = 1$, -1.

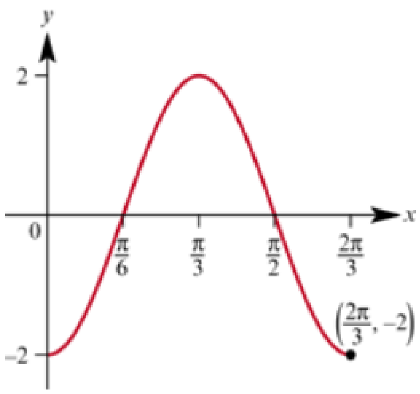
6 a The graph of $y = 2\cos 2x$ has period $\dfrac{2\pi}{2} = \pi$ and amplitude 2.



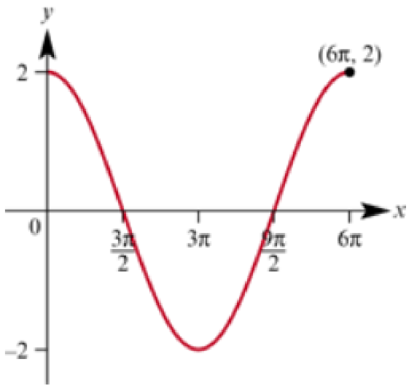
b The graph of $y = -3\sin\dfrac{x}{3}$ has period $\dfrac{2\pi}{\frac{1}{3}} = 6\pi$ and amplitude 3. It is $y = 3\sin\dfrac{x}{3}$ reflected in the x-axis.



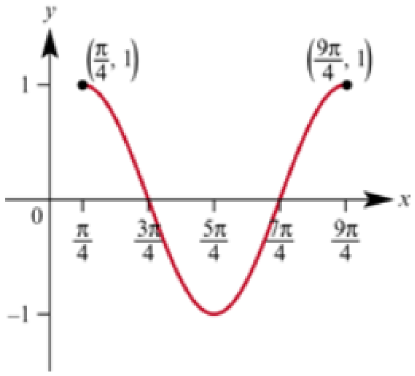
c The graph of $y = -2\cos 3x$ (period $\dfrac{2\pi}{3}$, amplitude 2), reflected in the x-axis.



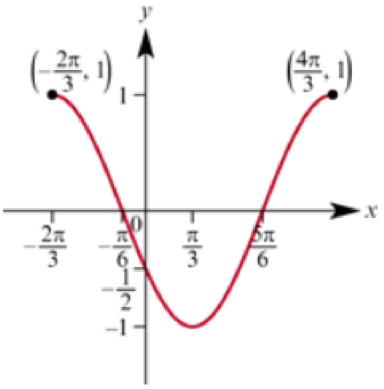
d The graph of $y = 2\cos\dfrac{x}{3}$ has period $\dfrac{2\pi}{\frac{1}{3}} = 6\pi$ and amplitude 3.



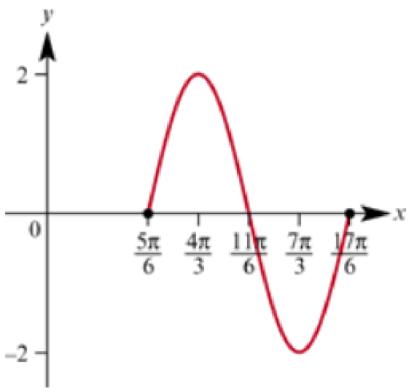
e The graph of $y = \cos x$, translated $\dfrac{\pi}{4}$ units to the right.



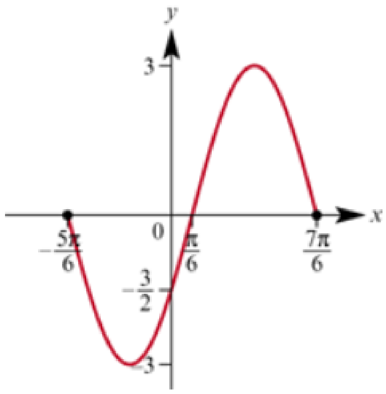
f The graph of $y = \cos x$, translated $\dfrac{2\pi}{3}$ units to the left.



g The graph of $y = 2\sin x$, translated $\dfrac{5\pi}{6}$ units to the right.



h The graph of $y = 3\sin x$, reflected in the x-axis and translated $\dfrac{\pi}{6}$ units to the left.



7 a $\cos\theta = -\frac{\sqrt{3}}{2}$

$\theta = \frac{5\pi}{6}$ and $-\frac{5\pi}{6}$

b $2\theta \in [-2\pi, 2\pi]$

$\cos 2\theta = \frac{\sqrt{3}}{2}$

$2\theta = -\frac{7\pi}{6}, -\frac{5\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$

$\theta = -\frac{7\pi}{12}, -\frac{5\pi}{12}, \frac{5\pi}{12}$ and $\frac{7\pi}{12}$

c $\theta - \frac{\pi}{3} \in \left[-\frac{\pi}{3}, \frac{5\pi}{3}\right]$

$\cos\left(\theta - \frac{\pi}{3}\right) = \frac{1}{2}$

$\theta = \pi$ and $\frac{5\pi}{3}$

d $\theta + \frac{\pi}{3} \in \left[\frac{\pi}{3}, \frac{7\pi}{3}\right]$

$\cos\left(\theta + \frac{\pi}{3}\right) = -1$

$\theta = \frac{2\pi}{3}$

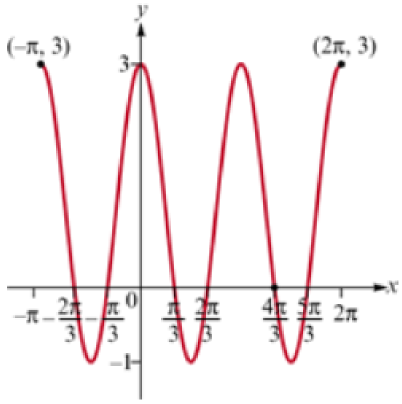
e

$\frac{\pi}{3} - \theta \in \left[-\frac{5\pi}{3}, \frac{\pi}{3}\right]$

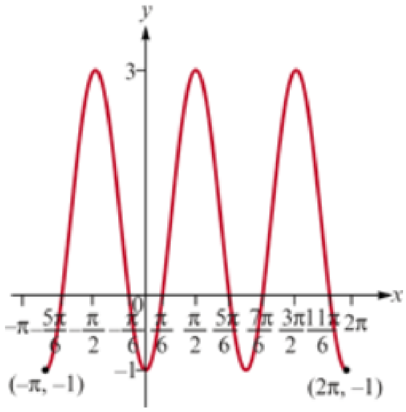
$\cos\left(\frac{\pi}{3} - \theta\right) = \frac{1}{2}$

$\theta = \pi$ and $\frac{5\pi}{3}$

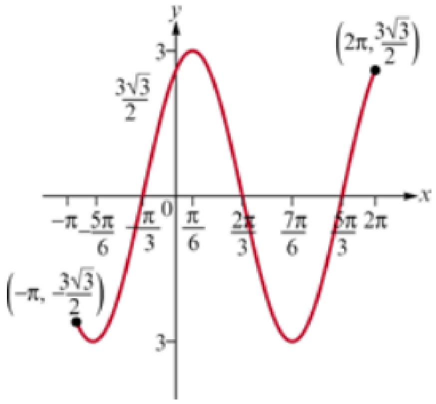
8 a The graph of $y = 2\cos 2x$ (period π, amplitude 2) translated 1 unit up.



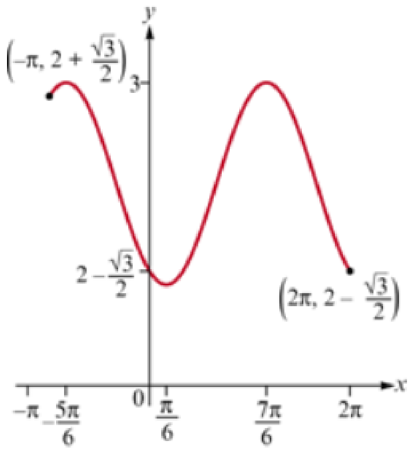
b The graph of $y = 2\cos 2x$ reflected in the x-axis and translated 1 unit up.



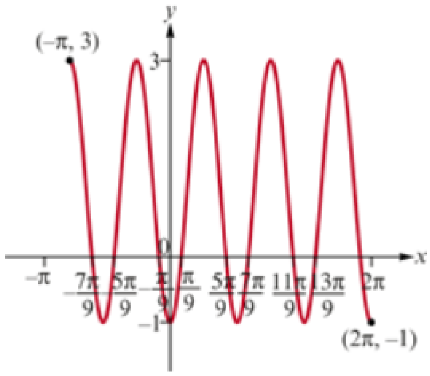
c The graph of $y = 3\sin x$ translated $\frac{\pi}{3}$ units to the left.



d The graph of $y = \sin x$ reflected in the x-axis and translated $\frac{\pi}{3}$ units to the left and 2 units up.



e The graph of $y = 2\cos 3x$ $\left(\text{period } \frac{2\pi}{3}, \text{ amplitude } 2\right)$ reflected in the x-axis and translated 1 unit up.



9 a

$$\tan x = -\sqrt{3}$$

$$\therefore x = \pi - \frac{\pi}{3}, 2\pi - \frac{\pi}{3}$$

$$\therefore x = \frac{2\pi}{3}, \frac{5\pi}{3} \text{ as } x \in [0, 2\pi]$$

b $\tan\left(3x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$

as $x \in [0, 2\pi]$

$$\therefore 3x \in [0, 6\pi]$$

$$\therefore 3x - \frac{\pi}{6} \in \left[-\frac{\pi}{6}, \frac{35\pi}{6}\right]$$

$$\therefore 3x - \frac{\pi}{6} = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{\pi}{6} + 2\pi, \frac{7\pi}{6} + 2\pi, \frac{\pi}{6} + 4\pi, \frac{7\pi}{6} + 4\pi$$

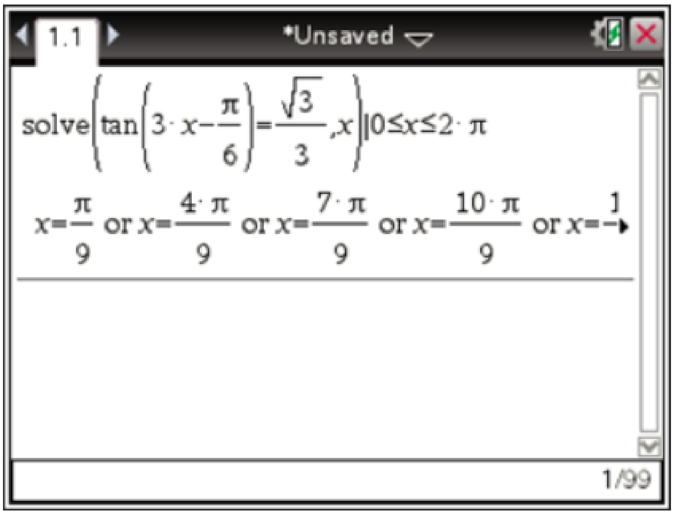
$$\therefore 3x - \frac{\pi}{6} = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{19\pi}{6}, \frac{25\pi}{6}, \frac{31\pi}{6}$$

$$\therefore 3x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \frac{10\pi}{3}, \frac{13\pi}{3}, \frac{16\pi}{3}$$

$$\therefore x = \frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, \frac{10\pi}{9}, \frac{13\pi}{9}, \frac{16\pi}{9}$$

CAS: Type

$$\mathbf{solve}\left(\mathbf{tan}\left(3x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}, x\right)\Big|0 \le x \le 2\pi$$



Use the right arrow key to view all solutions.

c

$$2\tan\left(\frac{x}{2}\right) + 2 = 0$$

$\therefore \tan\left(\frac{x}{2}\right) = -1$

and $\frac{x}{2} \in [0,\ \pi]$

$\therefore \frac{x}{2} = \frac{3\pi}{4}$ as $\frac{x}{2} \in [0,\ \pi]$

$\therefore x = \frac{3\pi}{2}$

d $3\tan\left(\frac{\pi}{2} + 2x\right) = -3$

$\therefore \tan\left(\frac{\pi}{2} + 2x\right) = -1$

as $x \in [0, 2\pi]$

$\therefore \frac{\pi}{2} + 2x \in \left[\frac{\pi}{2}, \frac{9\pi}{2}\right]$

$\therefore \frac{\pi}{2} + 2x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{3\pi}{4} + 2\pi, \frac{7\pi}{4} + 2\pi$

$\therefore \frac{\pi}{2} + 2x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}$

$\therefore 2x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$

$\therefore x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

10a $f(x) = \tan(2x)$

$\text{Period:} = \frac{\pi}{|n|} = \frac{\pi}{2}$

Asymptotes:

$x = \frac{(2k+1)\pi}{2n}$

$\therefore x = \frac{(2k+1)\pi}{4}$

$\therefore x = \frac{\pi}{4}, \frac{3\pi}{4}$ as $x \in [0,\ \pi]$

x-intercepts:

as $x \in [0,\ \pi]$

$\therefore 2x \in [0, 2\pi]$

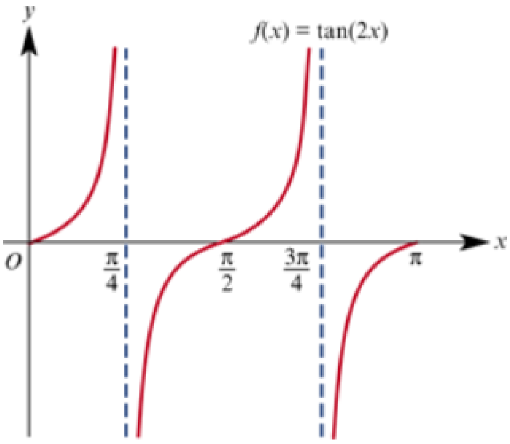
$\tan(2x) = 0$

$\therefore 2x = 0,\ \pi,\ 2\pi$

$\therefore x = 0,\ \frac{\pi}{2},\ \pi$

y-intercept:

$f(0) = \tan(0) = 0$



b

$f(x) = \tan\left(x - \frac{\pi}{3}\right)$

$\text{Period:} = \frac{\pi}{|n|} = \pi$

Asymptotes:

$x = \frac{(2k+1)\pi}{2n} + \frac{\pi}{3}$

$\therefore x = \frac{(2k+1)\pi}{2} + \frac{\pi}{3}$

$\therefore x = \frac{\pi}{2} + \frac{\pi}{3}$ as $x \in [0,\ \pi]$

$\therefore x = \frac{5\pi}{6}$

x-intercepts:

as $x \in [0,\ \pi]$

$\therefore x - \frac{\pi}{3} \in \left[-\frac{\pi}{3}, \frac{2\pi}{3}\right]$

$\tan\left(x - \frac{\pi}{3}\right) = 0$

$\therefore x - \frac{\pi}{3} = 0$

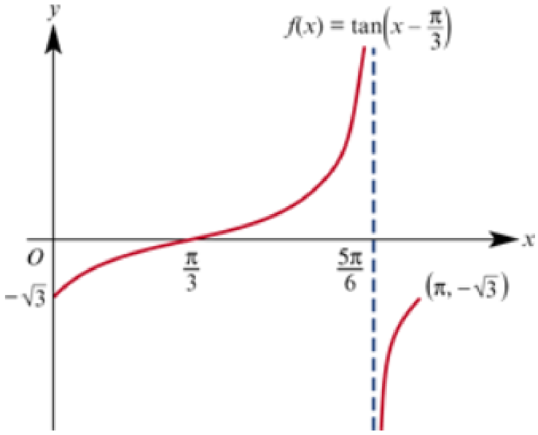
$\therefore x = \frac{\pi}{3}$

y-intercept:

$f(0) = \tan\left(-\frac{\pi}{3}\right) = -\tan\left(\frac{\pi}{3}\right) = -sqrt3$

Endpoint:

$f(\pi) = \tan\left(\frac{2\pi}{3}\right) = -\sqrt{3}$



c

$$f(x) = 2\tan\left(2x + \frac{\pi}{3}\right) = 2\tan\left(2\left(x + \frac{\pi}{6}\right)\right)$$

$$\text{Period: } = \frac{\pi}{|n|} = \frac{\pi}{2}$$

Asymptotes:

$$x = \frac{(2k+1)\pi}{2n} - \frac{\pi}{6}$$

$$\therefore x = \frac{(2k+1)\pi}{4} - \frac{\pi}{6}$$

$$\therefore x = \frac{\pi}{4} - \frac{\pi}{6}, \frac{3\pi}{4} - \frac{\pi}{6} \text{ as } x \in [0, \pi]$$

$$\therefore x = \frac{\pi}{12}, \frac{7\pi}{12}$$

x-intercepts:

as $x \in [0, \pi]$

$$\therefore 2x + \frac{\pi}{3} \in \left[\frac{\pi}{3}, \frac{7\pi}{3}\right]$$

$$\therefore \tan\left(2x + \frac{\pi}{3}\right) = 0$$

$$\therefore 2x + \frac{\pi}{3} = \pi, 2\pi$$

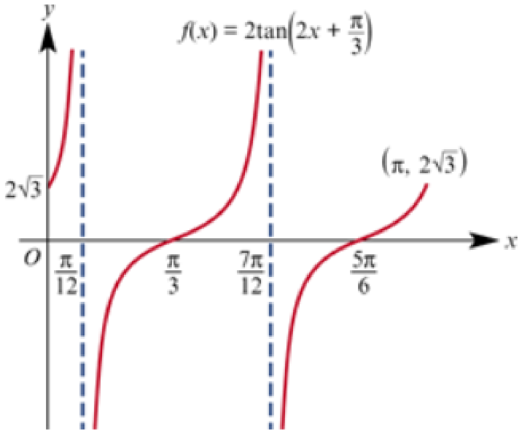
$$\therefore x = \frac{\pi}{3}, \frac{5\pi}{6}$$

y-intercept:

$$f(0) = 2\tan\left(\frac{\pi}{3}\right) = 2\sqrt{3}$$

Endpoint:

$$f(\pi) = 2\tan\left(\frac{7\pi}{3}\right) = 2\sqrt{3}$$



d

$$f(x) = 2\tan\left(2x + \frac{\pi}{3}\right) - 2$$
$$= 2\tan\left(2\left(x + \frac{\pi}{6}\right)\right) - 2$$

Period: $= \frac{\pi}{|n|} = \frac{\pi}{2}$

Asymptotes:

$$x = \frac{(2k+1)\pi}{2n} - \frac{\pi}{6}$$
$$\therefore x = \frac{(2k+1)\pi}{4} - \frac{\pi}{6}$$
$$\therefore x = \frac{\pi}{4} - \frac{\pi}{6}, \frac{3\pi}{4} - \frac{\pi}{6} \text{ as } x \in [0, \pi]$$
$$\therefore x = \frac{\pi}{12}, \frac{7\pi}{12}$$

x-intercepts:

as $x \in [0, \pi]$

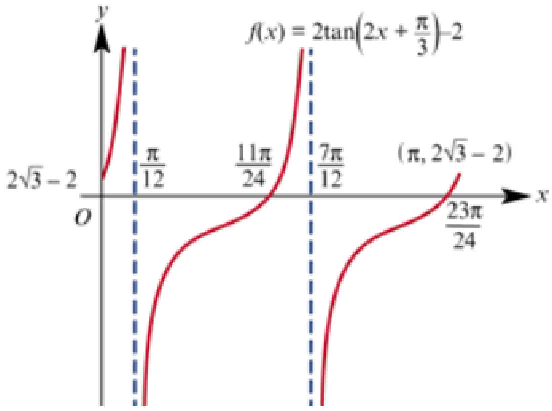
$$\therefore 2x + \frac{\pi}{3} \in \left[\frac{\pi}{3}, \frac{7\pi}{3}\right]$$
$$\therefore \tan\left(2x + \frac{\pi}{3}\right) = 1$$
$$\therefore 2x + \frac{\pi}{3} = \frac{5\pi}{4}, \frac{9\pi}{4}$$
$$\therefore x = \frac{11\pi}{24}, \frac{23\pi}{24}$$

y-intercept:

$$f(0) = 2\tan\left(\frac{\pi}{3}\right) - 2 = 2\sqrt{3} - 2$$

Endpoint:

$$f(\pi) = 2\tan\left(\frac{7\pi}{3}\right) - 2 = 2\sqrt{3} - 2$$



11a $\sin^2\theta = \frac{1}{4}$

$$\sin\theta = \pm\frac{1}{2}$$
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

b $\sin 2\theta = \frac{1}{2}$

$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$
$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

c $\cos 3\theta = \frac{\sqrt{3}}{2}$

$$3\theta = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}, \frac{25\pi}{6}, \frac{35\pi}{6}$$
$$\theta = \frac{\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{23\pi}{18}, \frac{25\pi}{18}, \frac{35\pi}{18}$$

d $\sin^2 2\theta = 1$

$$\sin 2\theta = \pm 1$$
$$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$
$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

12

$$\tan\theta = 2\sin\theta$$
$$\frac{\sin\theta}{\cos\theta} = 2\sin\theta$$
$$\frac{\sin\theta}{\cos\theta} - 2\sin\theta = 0$$
$$\sin\theta\left(\frac{1}{\cos\theta} - 2\right) = 0$$
$$\sin\theta = 0 \text{ or } \cos\theta = \frac{1}{2}$$
$$\theta = 60^\circ, 300^\circ, 0^\circ, 180^\circ, 360^\circ$$

13a $n\pi + \frac{\pi}{4}, n \in \mathbb{Z}$

b $\frac{2n\pi}{3}, n \in \mathbb{Z}$

c $-\frac{\pi}{4} + n\pi, n \in \mathbb{Z}$

Solutions to multiple-choice questions

1 A

$$\begin{aligned} y &= 2\sin\,(3x - \pi) + 4 \\ &= 2\sin 3\left(x - \frac{\pi}{3}\right) + 4 \end{aligned}$$

$$\text{Period} = \frac{2\pi}{3}$$

2 D $y = -5\cos 5x + 3$

$$\text{Amplitude} = |-5| = 5$$

3 D $5\sin\,(2x - \pi) + 2 = 0$

$$\begin{aligned} \text{Period} &= \frac{2\pi}{2} = \pi \\ \text{Amplitude} &= 5 \end{aligned}$$

Maximum and minimum y values are $y = 7$ and $y = -3$.

Since $y = 0$ is within the range of the function, the function will pass through the line twice each cycle. The function covers two cycles over the given domain, and so will pass through $y = 0$ four times.

There are four solutions.

4 C

$$\begin{aligned} \frac{3\pi}{11} &= \frac{3 \times \pi \times 180}{\pi \times 11} \\ &= \frac{540}{11} \\ &= 49.09 \end{aligned}$$

5 D

$$\begin{aligned} x &\in \left(\frac{5\pi}{12}, \frac{23\pi}{12}\right) \\ 3x &\in \left(\frac{5\pi}{4}, \frac{23\pi}{4}\right) \end{aligned}$$

$$\begin{aligned} \sin 3x &= -\frac{\sqrt{2}}{2} = -\frac{1}{\sqrt{2}} \\ 3x &= \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}, 3\pi + \frac{\pi}{4}, \\ &\quad 4\pi - \frac{\pi}{4}, 5\pi + \frac{\pi}{4}, 6\pi - \frac{\pi}{4} \end{aligned}$$

Omit the first and last as they are not in the interval.

$$\begin{aligned} 3x &= \frac{7\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}, \frac{21\pi}{4} \\ x &= \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{4}, \frac{7\pi}{4} \end{aligned}$$

6 E

$$\begin{aligned} \cos\left(-\frac{13\pi}{6}\right) &= \cos\left(-2\pi - \frac{\pi}{6}\right) \\ &= \cos -\frac{\pi}{6} \\ &= \sin\frac{2\pi}{3} \end{aligned}$$

7 **E** $\tan\ (180-\theta)^\circ = \dfrac{\sin\ (180-\theta)^\circ}{\cos\ (180-\theta)^\circ}$

$$= \frac{\sin\theta^\circ}{-\cos\theta^\circ}$$

$$= \frac{-\cos\ (90+\theta)^\circ}{-\sin\ (90+\theta)^\circ}$$

$$= \frac{\cos\ (90+\theta)^\circ}{\sin\ (90+\theta)^\circ}$$

8 **C** All other responses are $\cos x$

9 **D**

10 **B** Period of $4\sin(3\pi x)$ is $\dfrac{2}{3}$

Period of $3\sin(2\pi x)$ is 1

Period of the graph 2 since $3\times\dfrac{2}{3}=2$

Solutions to extended-response questions

1 **a** $\text{Maximum depth} = 15.4\text{ m}$ at midnight and noon.
$\text{Minimum depth} = 11.4\text{ m}$ at 6:00 and 18:00.

Time is measured in hours with $t=0$ corresponding to midnight on a particular day.

$$D(t) = a + b\cos\left(\frac{2\pi t}{k}\right)$$

i The 'centre' of the graph is given by

$$\frac{15.4+11.4}{2} = 13.4$$

Therefore $a = 13.4$.

ii $\text{The amplitude} = 13.4 - 11.4$
$= 2$

Therefore $b=2$.

iii $\text{Period} = 12\text{ hours}$ (the time between high tides).

$$\text{Therefore } 2\ \pi \div \frac{2\pi}{k} = 12$$

$$\text{and hence } k = 12$$

b When the depth of the water is 13.4 m

$$13.4 = 13.4 + 2\cos\left(\frac{\pi t}{6}\right)$$

$$\text{This implies} \cos\left(\frac{\pi t}{6}\right) = 0$$

$$\therefore\ \frac{\pi t}{6} = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \text{ or } \frac{5\pi}{2} \text{ or } \frac{7\pi}{2}$$

Hence $t = 3$ or 9 or 15 or21.

The depth of the water is 13.4 m at $3{:}00\text{ am}$, $9{:}00\text{ am}$, $3{:}00\text{ pm}$ and $9{:}00\text{ pm}$.

c Consider the inequality

$$13.4 + 2\cos\left(\frac{\pi t}{6}\right) < 14.4$$

which is equivalent to

$$\cos\left(\frac{\pi t}{6}\right) < \frac{1}{2}$$

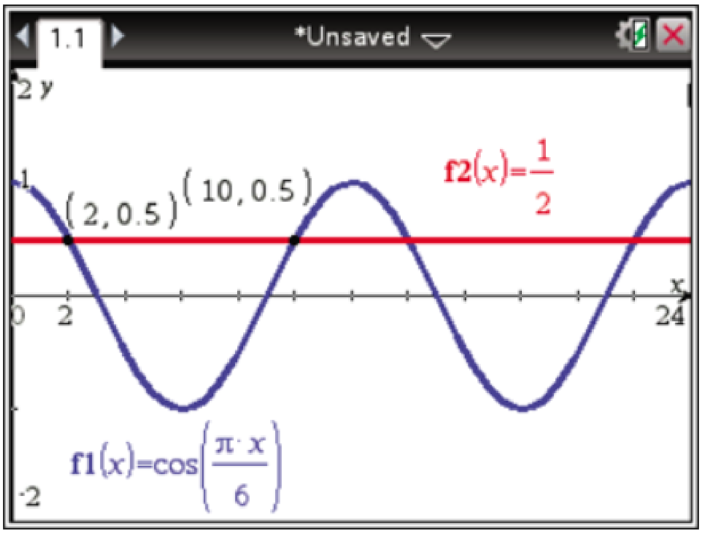
First consider the equation

$$\cos\left(\frac{\pi t}{6}\right) = \frac{1}{2}$$

$$\therefore \quad \frac{\pi t}{6} = \frac{\pi}{3} \text{ or } \frac{5\pi}{3} \text{ or } \frac{7\pi}{3} \text{ or } \frac{11\pi}{3}$$

$$\text{and } t = 2 \text{ or } 10 \text{ or } 14 \text{ or } 22$$

From the graph of $y = \cos\left(\frac{\pi t}{6}\right)$ this is true for $2 < t < 10$ or $14 < t < 22$.



2 $T = 15 - 8\cos\left(\frac{\pi t}{12} + 6\right)$ for $0 \le t \le 24$

a When $t = 0, T = 15 - 8\cos(6)$

$$= 7.3^\circ, \text{correct to two significant figures.}$$

b The maximum temperature $= 15 - (-8)$

$$= 23^\circ$$

The minimum temperature $= 15 - 8$

$$= 7^\circ$$

c Temperatures are warmer than 20° when

$$15 - 8\cos\left(\frac{\pi t}{12} + 6\right) > 20$$

Rearranging the inequality gives

$$\cos\left(\frac{\pi t}{12} + 6\right) < -\frac{5}{8}$$

Consider the equation

$$\cos\left(\frac{\pi t}{12} + 6\right) = -\frac{5}{8}$$

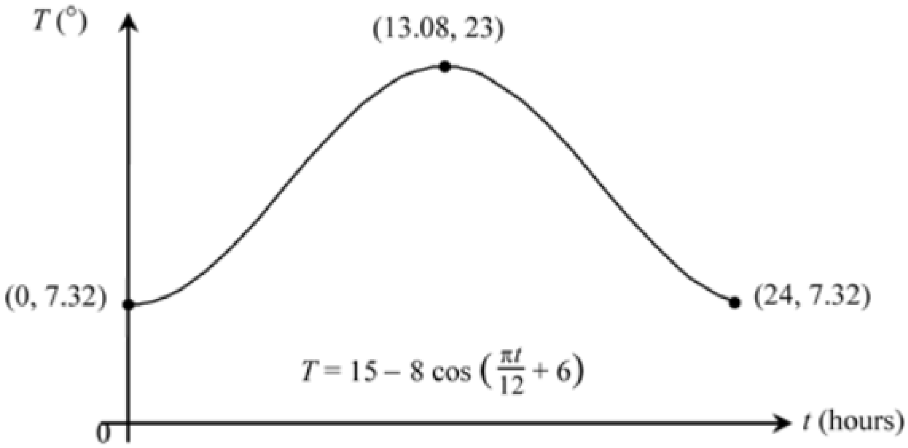
This gives $\quad \frac{\pi t}{12} + 6 = \pi - \cos^{-1}\left(\frac{5}{8}\right)$ or $\pi + \cos^{-1}\left(\frac{5}{8}\right)$ or $3\pi - \cos^{-1}\left(\frac{5}{8}\right)$ or $3\pi + \cos^{-1}\left(\frac{5}{8}\right)$

Solving for t where $0 \leq t \leq 24$ gives

$$t = 9.6605\ldots \text{ or } 16.5028\ldots$$

From the graph, temperatures are warmer than 20° between $9{:}40$ am and $4{:}30$ pm.

d



3 $x(t) = 3\sin\ (2\pi t - a)$

When $t = 1, x(t) = -1.5$

a To find a consider the equation

$$-1.5 = 3\sin\ (2\pi - a)$$

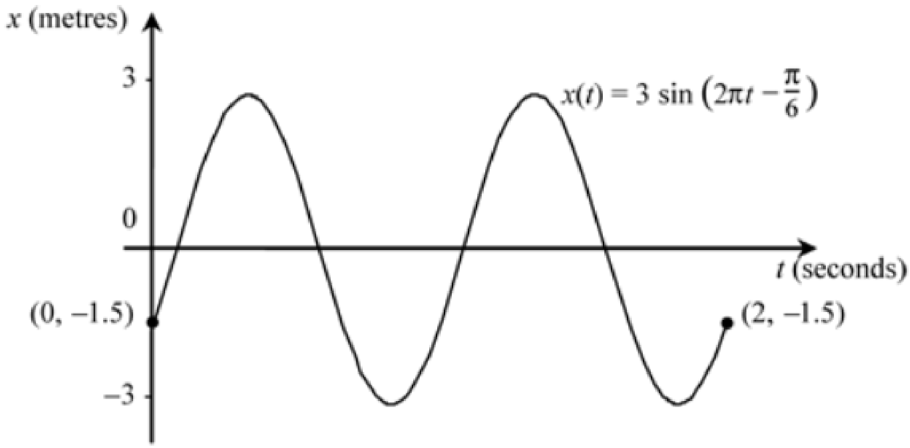
$$-\frac{1}{2} = \sin\ (2\pi - a)$$

$$-\frac{1}{2} = -\sin a$$

$$\text{Therefore } a = \frac{\pi}{6}$$

and $x(t) = 3\sin\left(2\pi t - \frac{\pi}{6}\right)$

b



c O is 3 metres from point A, as the amplitude of the graph is 3.

d When $x(t) = -3$,

$$3\sin\left(2\pi t - \frac{\pi}{6}\right) = -3$$

$$\therefore \quad \sin\left(2\pi t - \frac{\pi}{6}\right) = -1$$

$$\therefore \quad 2\pi t - \frac{\pi}{6} = \frac{3\pi}{2}$$

$$\therefore \quad 2\pi t = \frac{3\pi}{2} + \frac{\pi}{6}$$

$$\therefore \quad t = \frac{5}{6}$$

The particle first passes through A when $t = \frac{5}{6}$.

e The period of the motion is $\frac{2\pi}{2\pi} = 1$.

Therefore it takes 1 second to return to A.

f It will take $\frac{1}{4}$ of a period to go from A to O, i.e. $\frac{1}{4}$ second.

g **i** The first 2 seconds of motion is 2 periods. It will start at A and return to Aafter 1 second. From A to B and back is 12 metres. Therefore it will travel 24metres in 2 seconds. This can also be seen from the graph.

ii In 2.5 seconds it travels the 24 metres from part **g i** and an extra 6 metres, making a totaldistance of 30 metres.

4 $h(t) = p + q\sin\left(\frac{\pi t}{6}\right)$

a The maximum depth is 10.2 metres and the minimum depth is 1.8 metres.

The 'centre' is at $\frac{10.2 + 1.8}{2} = 6$.

Therefore $p = 6$.

The amplitude is $10.2 - 6 = 4.2$.

Therefore $q = 4.2$.

b The period of the function is $2\pi \div \frac{\pi}{6} = 12$

Maximum value of $6 + 4.2\sin\left(\frac{\pi t}{6}\right)$ occurs when

$$\sin\left(\frac{\pi t}{6}\right) = 1$$

$$\therefore \quad \frac{\pi t}{6} = \frac{\pi}{2} \text{ or } \frac{5\pi}{2}$$

$$\therefore \quad t = 3 \text{ or } 15$$

So the depth is a maximum at 3 am and 3 pm.

c The average depth is 6 metres for the 24 hour period.

d When $h(t) = 3.9, 3.9 = 6 + 4.2\sin\left(\frac{\pi t}{6}\right)$

Solve for t $\sin\left(\frac{\pi t}{6}\right) = -\frac{1}{2}$

$$\therefore \quad \frac{\pi t}{6} = \frac{7\pi}{6} \text{ or } \frac{11\pi}{6} \text{ or } \frac{19\pi}{6} \text{ or } \frac{23\pi}{6}$$

$$\therefore \quad t = 7 \text{ or } 11 \text{ or } 19 \text{ or } 23$$

So the depth of the water is 3.9 metres at 7 am, 11 am, 7 pm and 11 pm.

e Consider the inequation

$$6 + 4.2\sin\left(\frac{\pi t}{6}\right) > 8.1$$

$$\therefore \quad \sin\left(\frac{\pi t}{6}\right) > \frac{1}{2}$$

Consider the equation

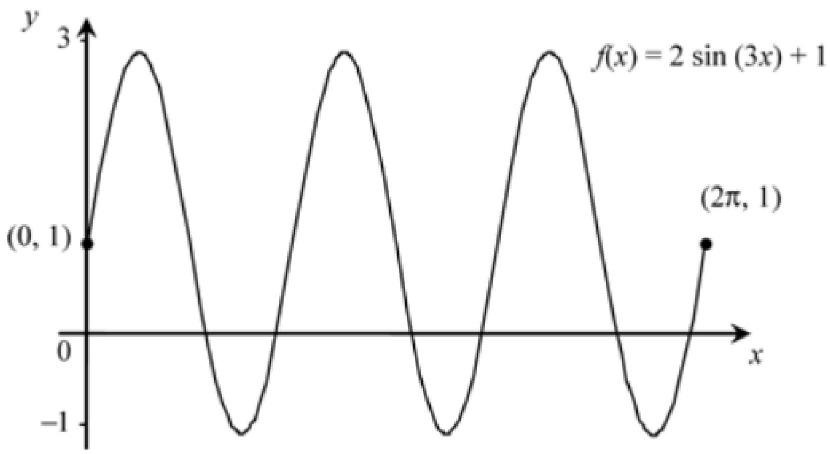
$$\sin\left(\frac{\pi t}{6}\right) = \frac{1}{2}$$

$$\therefore \quad \frac{\pi t}{6} = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{17\pi}{6}$$

$$\therefore \quad t = 1 \text{ or } 5 \text{ or } 13 \text{ or } 17$$

Therefore, from the graph, $\sin\left(\frac{\pi t}{6}\right) > \frac{1}{2}$ for the intervals $[1, 5]$ and $[13, 17]$. The duration of time for which the depth is greater than 8.1 m is 8 hours.

5



a **i** The line with equation $y = 1$ meets the graph 7 times. Therefore there are 6 intersections for $1 < k < 3$ or $-l < k < 1$.

ii The lines which pass through the maxima or minima give the threesolutions. Therefore $k = 3$ or $k = -1$.

iii There are no solutions for $k > 3$ or $k < -1$.

b The sequence of transformations which takes the graph of $y = 2\sin\ (3x) + 1$ to the graph of $y = \sin x$ can be found by rearranging the first of the equations.

i.e. $y = 2\sin\ (3x) + 1$

becomes $\frac{y-1}{2} = \sin\ (3x)$

and considering the equation of the image $y' = \sin x'$,

$$y' = \frac{y-1}{2}$$
$$= \frac{y}{2} - \frac{1}{2}$$
and $x' = 3x$

Therefore the sequence of transformations is a dilation of factor $\frac{1}{2}$ from the x axis and a dilation of factor 3 from the y axis followed by a translation of $\frac{1}{2}$ a unit in the negative direction of the y axis.

An alternative sequence is: a translation of 1 unit down,then a dilation of factor $\frac{1}{2}$ from the x-axis and a dilation of factor 3 from the y-axis.

c **i** For the graph of $y = f\ (x+h)$ to have a maximum at $\left(\frac{\pi}{3}, 3\right)$ consider

$$2\sin\ \left(3\left(\frac{\pi}{3}+h\right)\right)+1=3$$
$$\therefore\quad \sin\ (3h+\pi)=1$$
$$\text{which implies}\sin\ (3h)=-1$$
$$\text{and } 3h=\frac{3\pi}{2}\text{ or }\frac{7\pi}{2}$$
$$\therefore\quad h=\frac{\pi}{2}\text{ or }\frac{7\pi}{6}$$

ii For the graph of $y = f(x+h)$ to have a minimum at $\left(\frac{\pi}{3}, -1\right)$ consider

$$2\sin\ \left(3\left(\frac{\pi}{3}+h\right)\right)+1=-1$$
$$\therefore\quad \sin\ (3h+\pi)=-1$$
$$\text{which implies}\sin\ (3h)=1$$
$$\text{and } 3h=\frac{\pi}{2}\text{ or }\frac{5\pi}{2}$$
$$\therefore\quad h=\frac{\pi}{6}\text{ or }\frac{5\pi}{6}$$

6 **a** Consider $y = \sin x$

$$= \cos\ \left(\frac{\pi}{2}-x\right)$$
$$= \cos\ \left(x-\frac{\pi}{2}\right)$$

Therefore the graph of $y = \cos x$ can be taken to the graph of $y = \sin x$ by a translation of $\frac{\pi}{2}$ units in the positive direction of the x axis.

b Consider$y = -\frac{1}{2}\sin 2x$

$$= \frac{1}{2}\cos\ \left(2x+\frac{\pi}{2}\right)$$
$$= \frac{1}{2}\cos\ 2\ \left(x+\frac{\pi}{4}\right)$$

To find the sequence of transformations that take the graph of $y = 2\cos x$ to the graph of $y = -\frac{1}{2}\sin 2x$ compare

$\frac{y}{2} = \cos(x)$ and $2y' = \cos 2\left(x' + \frac{\pi}{4}\right)$

$$\text{Therefore } 2y' = \frac{y}{2}$$
$$\therefore \quad y' = \frac{y}{4}$$
$$\text{and } x = 2\left(x' + \frac{\pi}{4}\right)$$
$$\therefore \quad \frac{x}{2} = x' + \frac{\pi}{4}$$
$$\therefore \quad x' = \frac{x}{2} - \frac{\pi}{4}$$

The sequence of transformations is a dilation of factor $\frac{1}{2}$ from the y axis, then a translation of $\frac{\pi}{4}$ units in the negative direction of the x axis and a dilation of factor $\frac{1}{4}$ from the x axis.

c **i** $(x, y) \to \left(\frac{2}{\pi}x, 4 - y\right)$

$$x' = \frac{2}{\pi}x$$
$$\therefore \quad x = \frac{\pi x'}{2}$$
$$\text{and } y' = 4 - y$$
$$\therefore \quad y = 4 - y'$$

The graph of $y = \sin x$ is mapped to the graph of $4 - y' = \sin\left(\frac{\pi x'}{2}\right)$ and hence the image of $y = \sin x$ hasequation $y = 4 - \sin\left(\frac{\pi x}{2}\right)$.

ii The range is $[3, 5]$ and the period is 4.

7 $N(t) = 3000 \sin\left(\frac{\pi(t-1)}{6}\right) + 4000$

where t is the number of months after January 1.

$M(t) = 3000 \sin\left(\frac{\pi(t-3.5)}{5}\right) + 5500$

where t is the number of months after January 1.

a Maximum population of $N = 7000$ (occurs in April).
Maximum population of $M = 8500$ (occurs in October).
Minimum population of $N = 1000$ (occurs in June)
Minimum population of $M = 2500$ (occurs at the end of January and November)

b Sketch the graphs of

$$f1 = 3000 \sin\left(\frac{\pi(t-1)}{6}\right) + 4000 \text{ and}$$

$$f2 = 3000 \sin\left(\frac{\pi(t-3.5)}{5}\right) + 5500$$

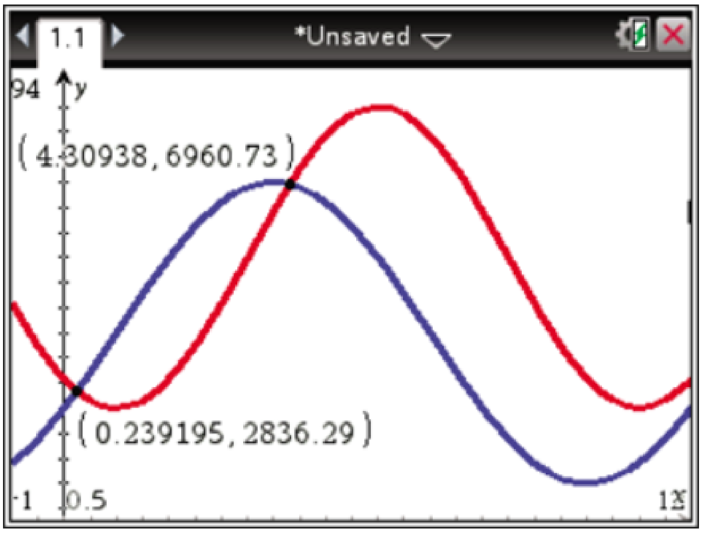
T1: Press **Menu** $\to$ **6:Analyze Graph** $\to$ **4:Intersection**

CP: Tap **Analysis** $\rightarrow$ **G-**

Solve $\rightarrow$ **Intersect**

The graphs cross when $t = 4.31$, i.e. populations are the same in April.

The population is then 6961. Also when $t = 0.24$, the populations are both 2836, i.e. the populations are the same in January.



c **T1:** Type $f1(x) + f2(x)$ into $f3(x) =$

Press Menu $\rightarrow$ **6:Analyze**

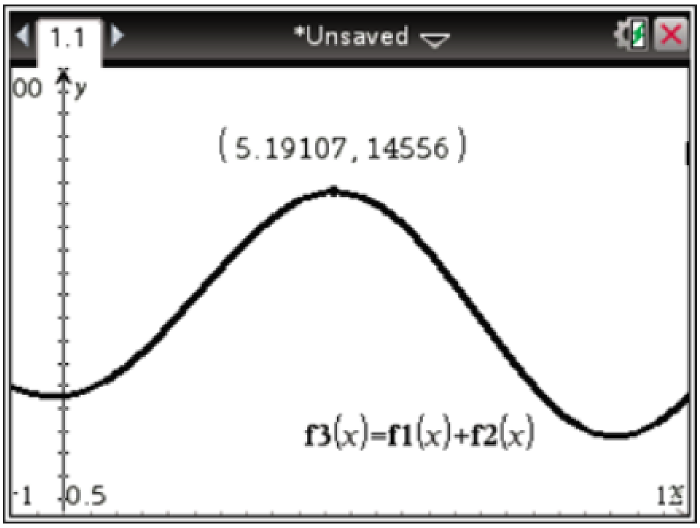
Graph $\rightarrow$ **3:Maximum**

CP: Type $y1(x) + y2(x)$ into $y3 =$

Tap **Analysis** $\rightarrow$ **G-Solve** $\rightarrow$ **Max**

The graph is as shown.

The maximum value of $y = N(t) + M(t)$ is $14\,556.039$ and this occurs when $t = 5.191$, i.e. in May



d **T1:** Type $f1(x) - f2(x)$ into $f3(x) =$

Press Menu $\rightarrow$ **6: Analyze**

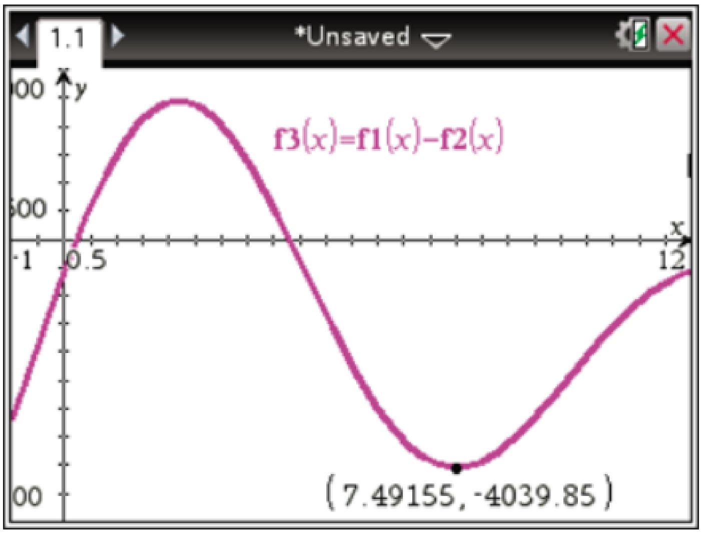
Graph $\rightarrow$ **2: Minimum**

CP: Type $y1(x) - y2(x)$ into $y3 =$

Tap **Analysis** $\rightarrow$ **G-Solve** $\rightarrow$ **Min**

The maximum difference is given by the local minimum.

The difference is 4039.845 and this occurs when $t = 7.49$, i.e. in July

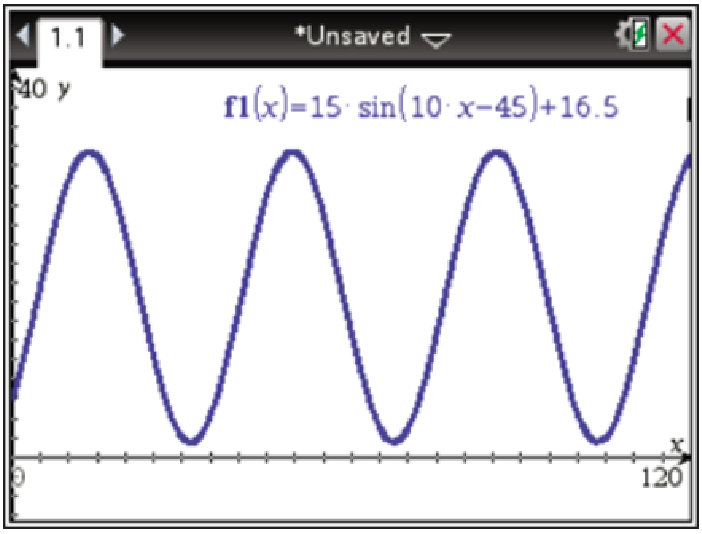


8 **a** $h = 15\sin\ (10t - 45)^{\circ} + 16.5$

T1: Change the Graphing Angle to Degree before sketching. Press

Menu $\rightarrow$ **9:Settings**

CP: Set to Deg mode before sketching.



b $\text{When } t = 0, h = 15\sin\ (-45)^{0} + 16.5$

$$= -15 \times \frac{1}{\sqrt{2}} + 16.5$$

$$= 5.89$$

c
$$5 = 15\sin(10t-45)^\circ + 16.5$$
$$-11.5 = 15\sin(10t-45)^\circ$$
$$\frac{-11.5}{15} = \sin(10t-45)^\circ$$
$$10t-45 = 180+\sin^{-1}\left(\frac{23}{30}\right) \text{ or } 360-\sin^{-1}\left(\frac{23}{30}\right)$$
$$\text{or } 540+\sin^{-1}\left(\frac{23}{30}\right) \text{ or } 720-\sin^{-1}\left(\frac{23}{30}\right)$$
$$10t = 225+\sin^{-1}\left(\frac{23}{30}\right) \text{ or } 405-\sin^{-1}\left(\frac{23}{30}\right)$$
$$\text{or } 585+\sin^{-1}\left(\frac{23}{30}\right) \text{ or } 765-\sin^{-1}\left(\frac{23}{30}\right)$$
$$t = \frac{1}{10}\left(225+\sin^{-1}\left(\frac{23}{30}\right)\right) \text{ or } \frac{1}{10}\left(405-\sin^{-1}\left(\frac{23}{30}\right)\right)$$
$$\text{or } \frac{1}{10}\left(585+\sin^{-1}\left(\frac{23}{30}\right)\right) \text{ or } \frac{1}{10}\left(765-\sin^{-1}\left(\frac{23}{30}\right)\right)$$

Isobel's seat passes the platform for the first time after

$$\frac{1}{10}\left(225+\sin^{-1}\left(\frac{23}{30}\right)\right) \text{ seconds } \approx 27.51 \text{ seconds.}$$

This can, of course, be found using **Intersection/Intersect** from a CAS calculator.

d Her seat will pass the platform six times in the first two minutes.

e The ride is six minutes long. The seat will pass the access platform 20 times in the entire ride.

f When $t = 100, h = 4.213$

Isobel is stranded 4.21 metres above the ground, correct to two decimal places.

g The position of Hamish's seat above the ground is given by
$$h_2 = 15\sin(10t-\alpha)^\circ + 16.5$$
$$\text{When } t = 0, h_2 = 1.5$$
$$-1 = \sin(-\alpha)^\circ$$
$$\alpha = 90$$
$$\therefore\ h_2 = 15\sin(10t-90)^\circ + 16.5$$
$$\text{When} t = 100, h_2 = 15\sin(1000-90)^\circ + 16.5 = 13.89$$

Hamish's seat was 13.9 metres above the ground, correct to one decimal place.